Lab 4: Frequency Domain Circuits

Key Concepts:

- Decibels (dB)
- Steady-state signals; Fourier series
- Complex impedances
- Low-pass and high-pass filters
- Frequency response
- Time domain vs. frequency domain
- Fourier transform, Fast Fourier Transform (FFT)

Decibels

Ratio = P_2/P_1 (linear)

Ratio (dB) = $10\log_{10}(P_2/P_1)$ (dB)

$P_{out} = 10\,P_{in} \Rightarrow$ Gain $= 10\log_{10}\left(\frac{10\,P_{in}}{P_{in}}\right) = 10$ dB

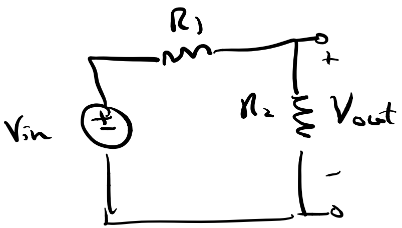
$P_{out} = 0.1\,P_{in} \Rightarrow$ Gain $= 10\log_{10}\left(\frac{0.1\,P_{in}}{P_{in}}\right) = -10$ dB

$P_{out} = P_{in}$ $\quad = 10\log_{10}(1) = 0$ dB

$P_{out} = 0.5\,P_{in} \Rightarrow$ $\quad = 10\log_{10}(0.5) = -3$ dB

$\log x^n = n\log x$

$\log x^2 = 2\log x$



$P = V^2/R$

$P_{out} = V_{out}^2/R$

$P_{in} = V_{in}^2/R$

$\Rightarrow 10\log_{10}\left(\frac{V_{out}^2/R}{V_{in}^2/R}\right) = 10\log_{10}\left(\frac{V_{out}^2}{V_{in}^2}\right)$

$= 20\log_{10}\left(\frac{V_{out}}{V_{in}}\right)$

$\frac{V_{out}}{V_{in}}$ (dB)?

Decibels cont.

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$$\frac{P_{max}}{P_{min}} = \text{Dynamic Range (linear)}$$

$$10 \log_{10}\left(\frac{P_{max}}{P_{min}}\right) = DR\ (dB)$$

$$20 \log_{10}\left(\frac{V_{max}}{V_{min}} = \frac{10\ V}{0.1\ mV}\right) = 100\ dB \rightarrow 5 \text{ orders of magnitude}$$

dBW : decibels relative to 1W : $10 \log_{10} \frac{P_1 \nearrow 100W}{1W} = 20\ dBW$

dBV : dB relative to 1V : $20 \log_{10}\left(\frac{V_1 \rightarrow 10V}{1V}\right) = 20\ dBV$

dBV_{rms}

dBm : dB relative to 1 mW : $P = 1W$

$$= 10 \log_{10} \frac{1W}{1\ mW} = 30\ dBm$$

Steady-state signals

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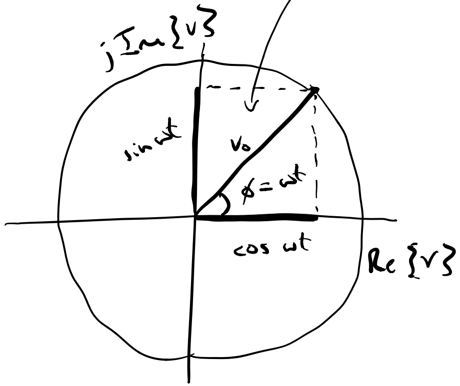
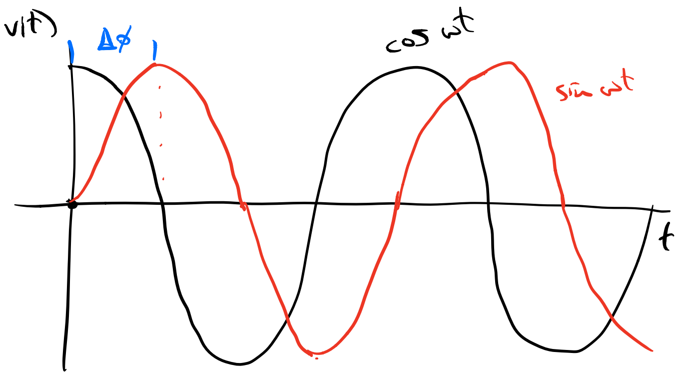
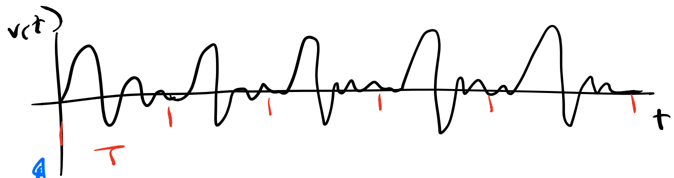
– AC source ; circuit is turned on , allowed to "settle" to "steady state"

$$\text{signal} = v(t) = \underbrace{V_o e^{j\omega t}} = V_o(\cos \omega t + j \sin \omega t) \checkmark$$





Fourier Series



Any periodic signal can be decomposed into a sum of sinusoids

?

$$v(t) = a_0 + \sum_{i=1}^{\infty} \left(a_i \cos \omega_i t + b_i \sin \omega_i t \right) = a_0 + \sum_{i=1}^{\infty} c_i e^{(j\omega_i t + \phi_i)}$$

$i = 2, 3, 4$

a_0 = DC component offset

ω_i = frequencies $= i \dfrac{2\pi}{T}$

$$c_i = \sqrt{a_i^2 + b_i^2}$$

$$\phi_i = -\tan^{-1}\left(\frac{b_i}{a_i}\right)$$

$v(t) \rightarrow a_0, a_i, b_i$

$$a_0 = \frac{1}{T}\int_0^T v(t)\, dt$$

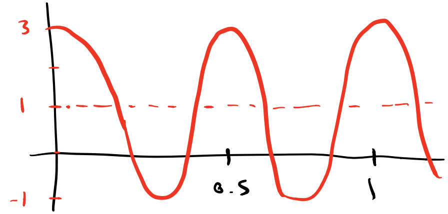
$$a_i = \frac{2}{T}\int_0^T v(t) \cos \omega_i t \, dt \qquad b_i = \frac{2}{T}\int_0^T v(t) \sin \omega_i t \, dt$$

Fourier Series Example 1

$$v(t) = \underline{1} + 2\cos(\underbrace{4\pi}t) \quad \checkmark$$

$$\omega = 4\pi$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \underline{0.5}$$



$$a_0 = \frac{1}{T}\int_0^T v(t)\,dt = 1$$

$$\omega_i = i\frac{2\pi}{T}$$

$$a_i = \frac{2}{T}\int_0^T (1 + 2\cos 4\pi t)(\cos \omega_i t)\,dt$$

$$a_1 = \frac{2}{T}\int_0^T (1 + 2\cos 4\pi t)\cos 4\pi t\,dt = 2$$

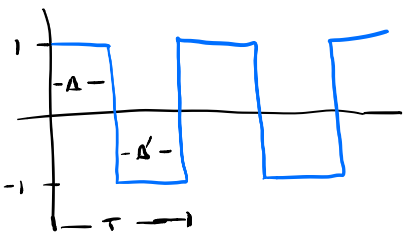
$$b_1 = \frac{2}{T}\int_0^T (1 + 2\cos 4\pi t)\sin 4\pi t\,dt = 0$$

$$v(t) = a_0 + \sum_{i=1}^{\infty}(a_i \cos\omega_i t + b_i \sin\omega_i t)$$

$$\underline{a_2, b_2, a_3, b_3, \ldots} = 0$$

$$= \underline{1 + 2\cos 4\pi t} + 0\sin 4\pi t + 0\cos 8\pi t + 0\sin 8\pi t \ldots$$

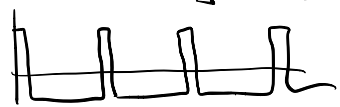
Example 2: Square wave



$\Delta = 50\%$

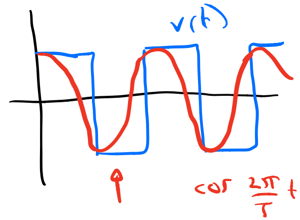
$\underline{a_0 = 0}$

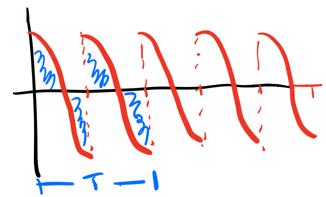
$\Delta = 10\%$



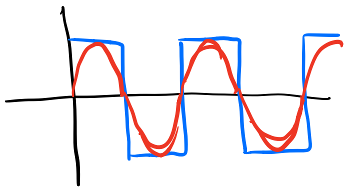
$$a_1 = \frac{2}{T}\int_0^T v(t)\cos\left(\frac{2\pi}{T}t\right)dt$$

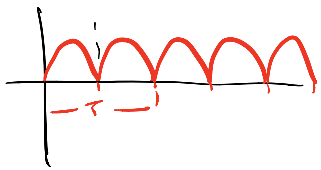
$$= 0 \qquad \underline{a_i = 0 \text{ for all } i}$$





$$b_1 = \frac{2}{T}\int_0^T v(t)\sin\frac{2\pi}{T}t\,dt$$





$$b_1 = \frac{2}{T}\left[2\int_0^{T/2}\sin\frac{2\pi}{T}t\,dt\right] = \frac{4}{\pi}$$

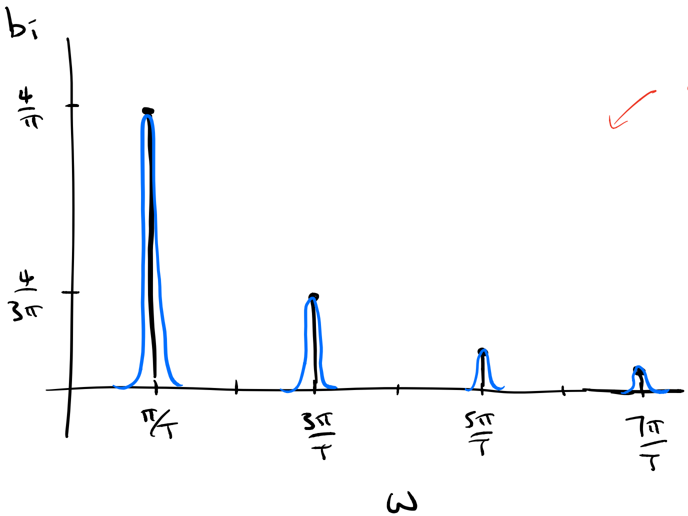
$$\underline{b_i = 0 \text{ for all even } i}$$

Square Wave cont.

General solution

$$v(t) = \frac{4}{\pi} \sum_{i=odd}^{\infty} \frac{1}{i} \sin\left(\frac{i\pi t}{T}\right)$$

b_i



discrete in f.

Fourier Series

- Square wave is sum of odd harmonics
- More harmonics included - closer to the truth!

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Fourier Series--Square Wave

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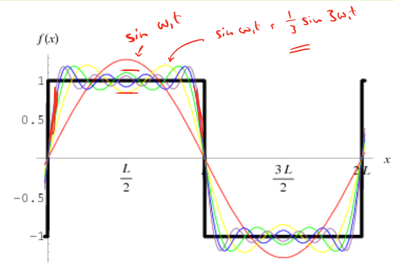


Consider a square wave $f(x)$ of length $2L$. Over the range $[0, 2L]$, this can be written as

$$f(x) = 2[H(x/L) - H(x/L - 1)] - 1, \tag{1}$$

where $H(x)$ is the Heaviside step function. Since $f(x) = f(2L - x)$, the function is odd, so $a_0 = a_n = 0$, and

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{2}$$

reduces to

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx & (3)\\ &= \frac{4}{n\pi}\sin^2\left(\tfrac{1}{2}n\pi\right) & (4)\\ &= \frac{2}{n\pi}[1-(-1)^n] & (5)\\ &= \frac{4}{n\pi}\begin{cases}0 & n \text{ even}\\ 1 & n \text{ odd.}\end{cases} & (6)\end{aligned}$$

The Fourier series is therefore

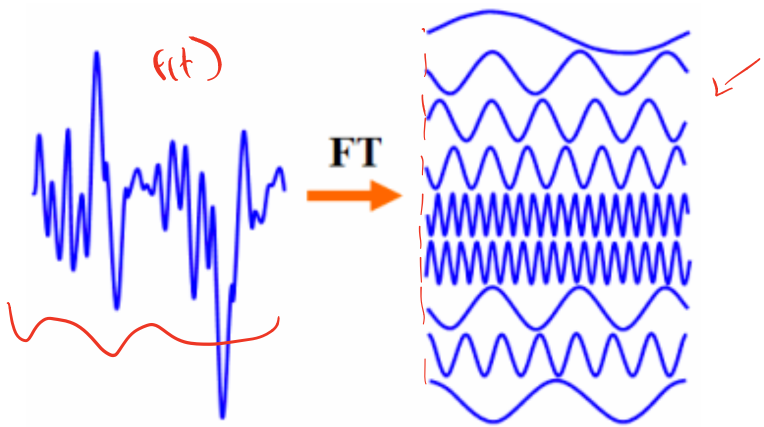
$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right). \tag{7}$$

Fourier Transform

- Fourier transform is sort-of the continuous (in frequency!) version of Fourier series; rather than sum, it uses integral
- Operates on **any function** (Fourier series only works for periodic functions)
- Fourier transform gives spectrum - separates signal into all of its component frequencies!
- Determined the magnitude and phase of each component (i.e., each frequency, from f = -infty to infty)

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$



Fourier Transforms

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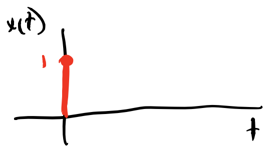
Fourier transform of any signal $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

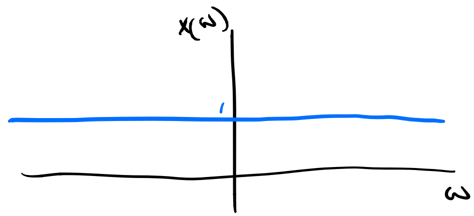
if $x(t)$ is in volts, $X(\omega)$ is in units of $V \cdot s$ or V/Hz

inverse FT: $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{+j\omega t}\, dt$$

example 1: $x(t) = \delta(t)$



$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$ because $\delta(t) = \begin{cases} 1, & t=0 \\ 0, & \text{elsewhere} \end{cases}$



More examples

example 2: $x(t) = 1$

$$X(\omega) = \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}\, dt = \delta(\omega), \quad \text{because} \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = \begin{cases} 1, & \omega = 0 \\ 0, & \text{elsewhere} \end{cases}$$

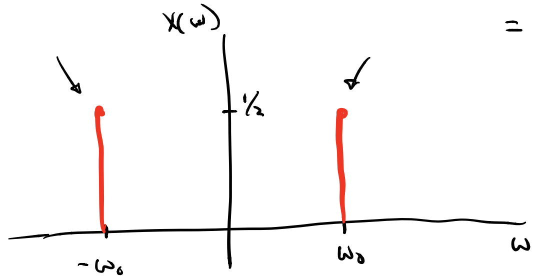
$\delta(t) \Longleftrightarrow 1$ form a fourier transform pair.

example 3: $x(t) = \cos \omega_0 t$

$$X(\omega) = \int_{-\infty}^{\infty} \cos \omega_0 t \; e^{-j\omega t}\, dt$$

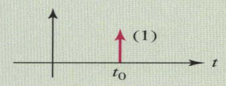
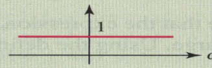
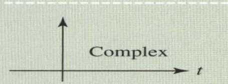
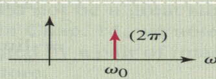
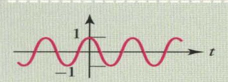
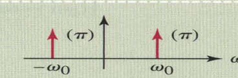
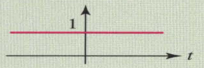
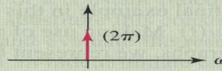
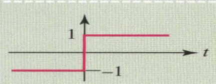
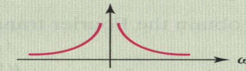
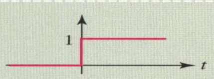
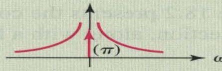
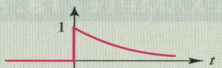
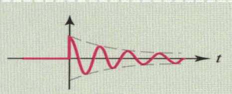
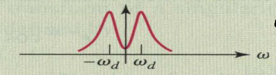
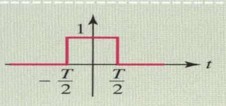
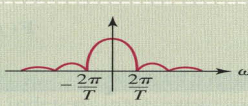
$$= \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right) e^{-j\omega t}\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{j(\omega - \omega_0)t} + e^{-j(\omega_0 + \omega)t} \right) dt$$

$$= \frac{1}{2} \left(\delta(\omega_0 - \omega) + \delta(\omega_0 + \omega) \right)$$



Fourier transform pairs

TABLE 18.2 A Summary of Some Fourier Transform Pairs

$f(t)$	$f(t)$	$\mathscr{F}\{f(t)\} = F(j\omega)$	$\lvert F(j\omega)\rvert$
	$\delta(t - t_0)$	$e^{-j\omega t_0}$	
	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
	$\cos\omega_0 t$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$	
	1	$2\pi\delta(\omega)$	
	$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$	
	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	
	$e^{-\alpha t}u(t)$	$\frac{1}{\alpha + j\omega}$	
	$[e^{-\alpha t}\cos\omega_d t]u(t)$	$\frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_d^2}$	
	$u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$	$T\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$	

Fourier Transform of derivative:

Fourier transform of $\frac{dx(t)}{dt}$:

if $\mathcal{F}(x(t)) \to X(\omega)$,

$$\mathcal{F}\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \frac{dx}{dt} e^{-j\omega t}\, dt$$

integrate by parts $\int u\, dv = uv - \int v\, du$

$u = e^{-j\omega t}$

$dv = \frac{dx}{dt} \cdot dt$, $v = x$, $du = -j\omega e^{-j\omega t}\, dt$

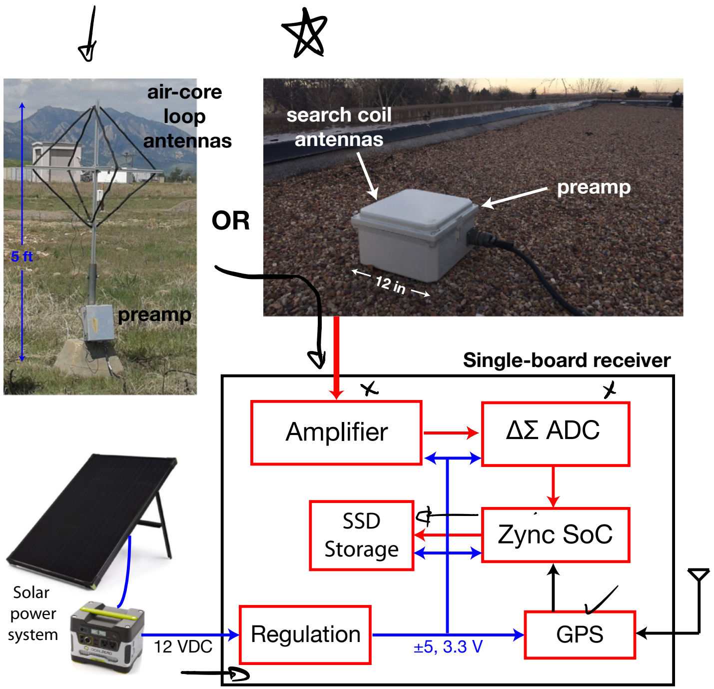
$$= \underbrace{x(t)\, e^{-j\omega t}\Big|_{t=-\infty}^{\infty}}_{v \cdot u} - \underbrace{\int_{-\infty}^{\infty} x(t)\left(-j\omega e^{-j\omega t}\right) dt}_{v\, du}$$

assuming $x(t) \to 0$ as $t \to \pm\infty$

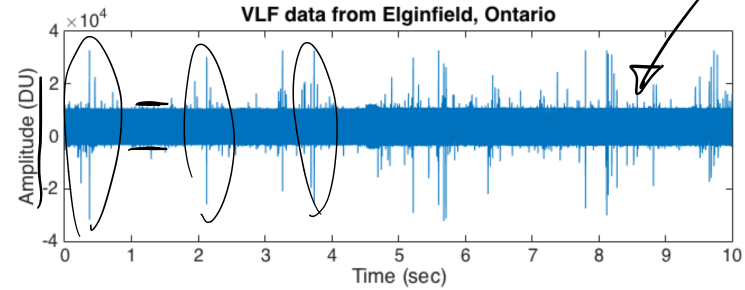
$$= 0 + j\omega \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}_{X(\omega)}$$

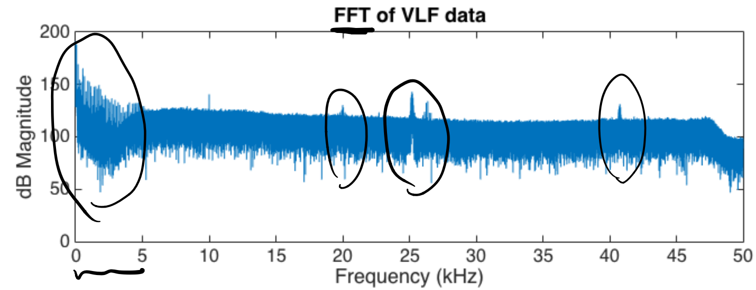
$$= j\omega X(\omega)$$

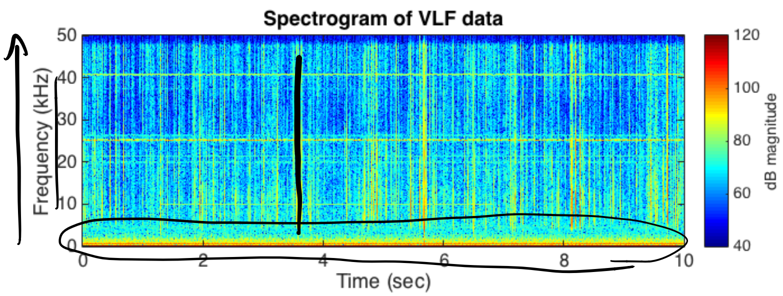
A real Fourier transform



time domain B-field







Comparing Fourier Series and Transform

FS: $x(t) = a_0 + \sum_{i=1}^{\infty} c_i e^{j\omega t} = a_0 + \sum_{i=1}^{\infty} (a_i \cos \omega_i t + b_i \sin \omega_i t)$

volts

domain: discrete frequencies

units: volts

signal: must be periodic

FT: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

domain: continuous in freq

units: V/Hz or V·S

signal: any

Comparing Laplace and Fourier Transform

LT: $X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$

$s = \sigma + j\omega$

$\mathrm{Re}\{s\}$ $\quad$ $\mathrm{Im}\{\omega\}$

units
domain
signal } same

FT: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

Discrete Fourier Transform and Fast Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

DFT: $$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N}$$

N = total number of samples

n = index in time of x

k = index in freq of X

FFT = Fast Fourier Transform

Complex Impedances

if my signal is $v(t) = V_0 e^{j\omega t}$, $i(t) = I_0 e^{j\omega t}$

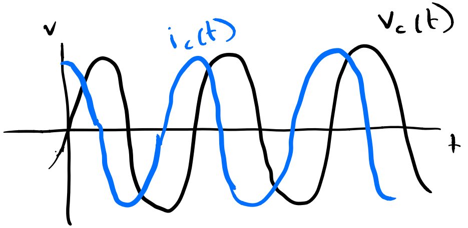
Resistor:

$$v(t) = i(t) \cdot R$$

$$V(\omega) = I(\omega) \cdot R$$

$$\frac{V(\omega)}{I(\omega)} = Z_R(\omega) = R$$

capacitor: current leads voltage by 90°

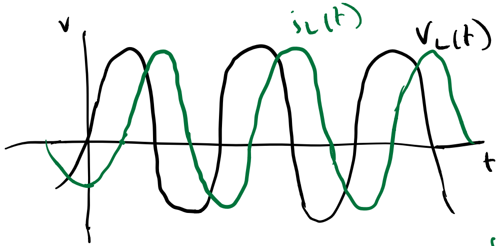


Capacitor:

$$i(t) = C \frac{dv(t)}{dt}$$

$$I(\omega) = C \cdot j\omega V(\omega)$$

$$\frac{V(\omega)}{I(\omega)} = Z_C(\omega) = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$



inductor: current lags voltage by 90°

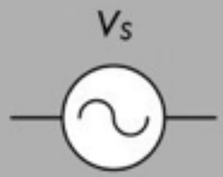
Inductor:

$$v(t) = L \frac{di(t)}{dt}$$

$$V(\omega) = L \cdot j\omega I(\omega)$$

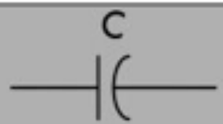
$$\frac{V(\omega)}{I(\omega)} = Z_L(\omega) = j\omega L$$

Complex Impedances



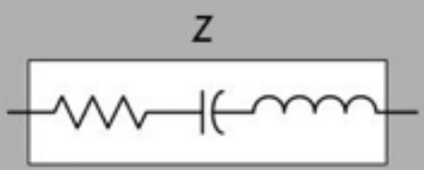
$V_S\angle 0°$

$R\angle 0°$



$X_C = -j\frac{1}{\omega C}, \quad X_C = \frac{1}{\omega C}\angle -90°$

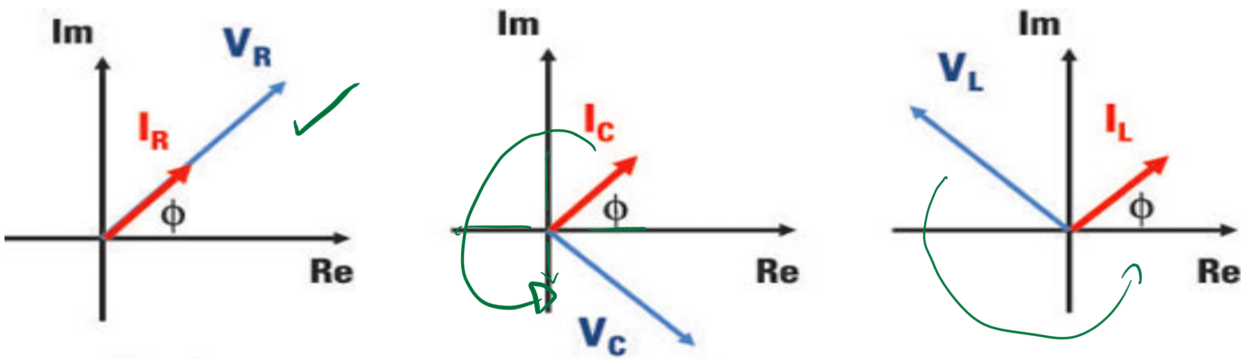
$X_L = j\omega L, \quad X_L = \omega L\angle +90°$



$$Z = R + X_C + X_L$$

$$= R - j\frac{1}{\omega C} + j\omega L = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + \left(\omega L - \frac{1}{\omega C}\right)\angle 90°$$

Complex Impedance Phase delay



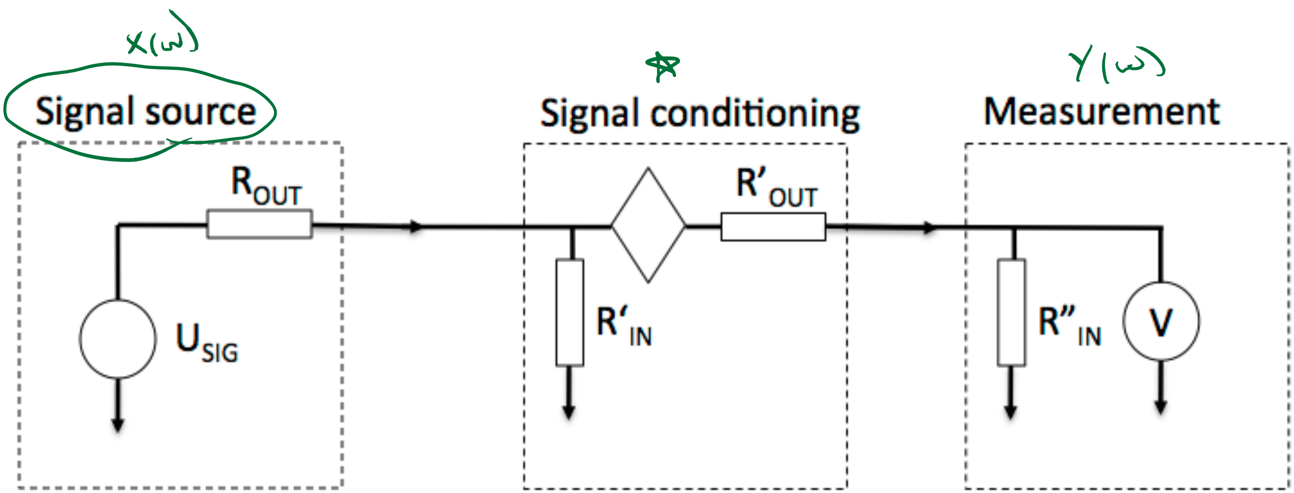
Resistor
Voltage in phase with current

Capacitor
Voltage lags current by 90°

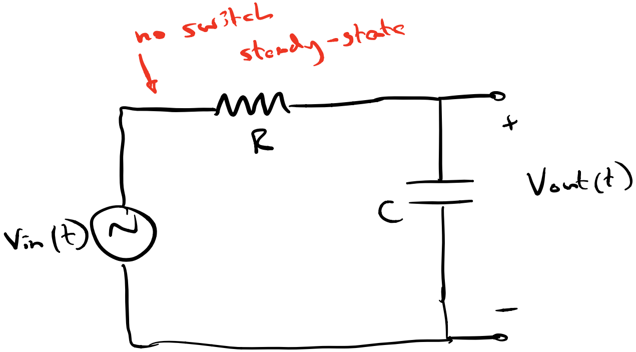
Inductor
Voltage leads current by 90°

Signal Conditioning (again)

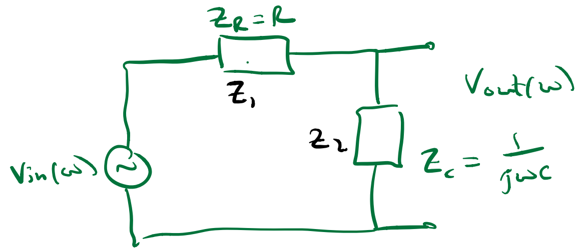
- Level shifting
- Gain / attenuation
- **Filtering**
- Buffering
- Impedance conversion



Voltage Divider



$\Rightarrow$



$$V_{out}(\omega) = V_{in}(\omega) \cdot \left[\frac{Z_2}{Z_1 + Z_2} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} \right]$$

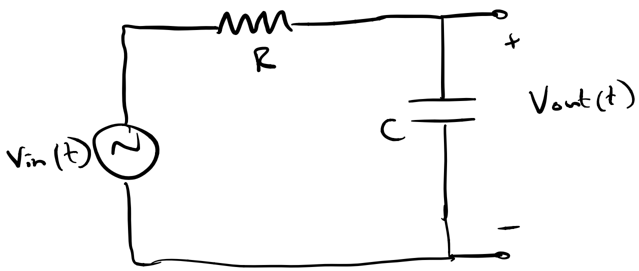
$$\frac{V_{out}(\omega)}{V_{in}(\omega)} = H(\omega) = \frac{1}{1 + j\omega RC}$$ is the transfer function.

$$H(\omega) = |H(\omega)|\, e^{j\phi(\omega)}$$

$$|H(\omega)| = \left(H(\omega) \cdot H^*(\omega) \right)^{1/2}$$ magnitude response

$$\phi(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}} \right)$$ phase response

Voltage Divider cont. $\Rightarrow$ Magnitude Response.



$$\left|\frac{V_{out}(\omega)}{V_{in}(\omega)}\right| = |H(\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$\phi(\omega) = -\tan^{-1}(\omega RC)$$

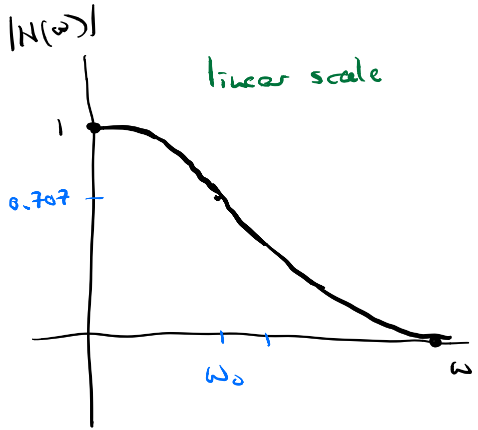
$2\pi f$

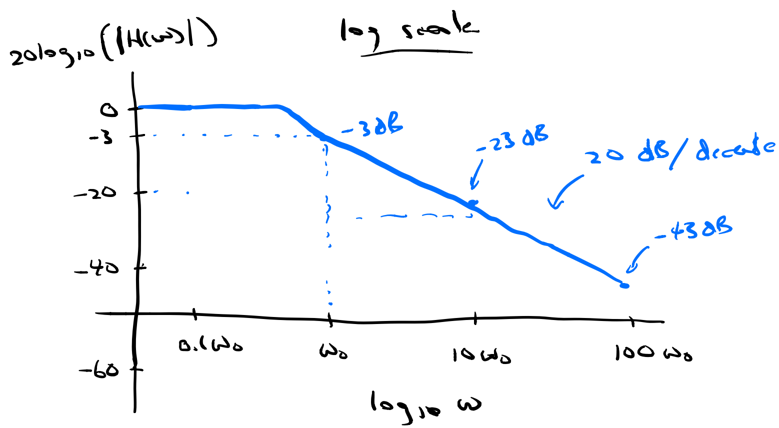
at $\omega = 0$, $|H(\omega)| = 1$

at $\omega \to \infty$, $|H(\omega)| = 0$

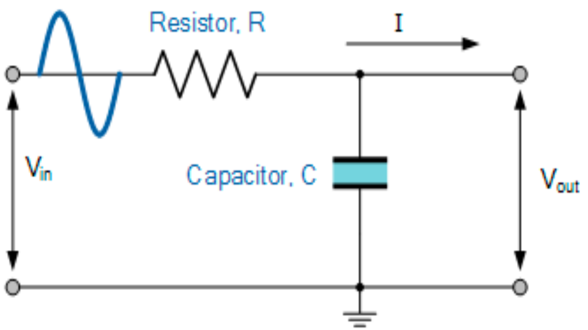
at $\omega_0 = 1/RC$, $|H(\omega)| = 1/\sqrt{2}$

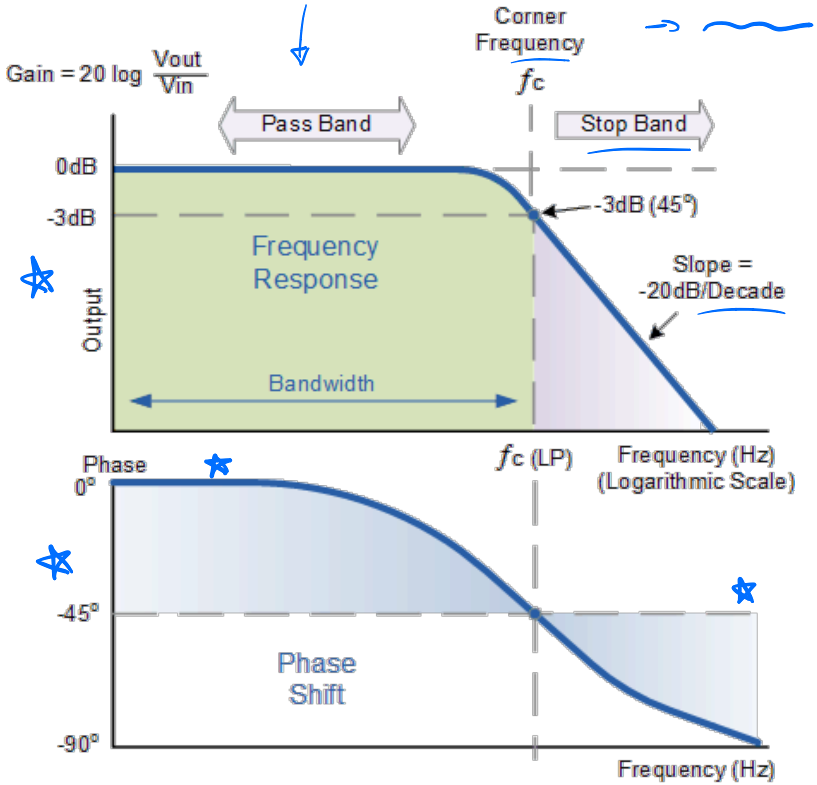
ω_0 = cutoff freq. OR −3 dB frequency





Low-pass filter

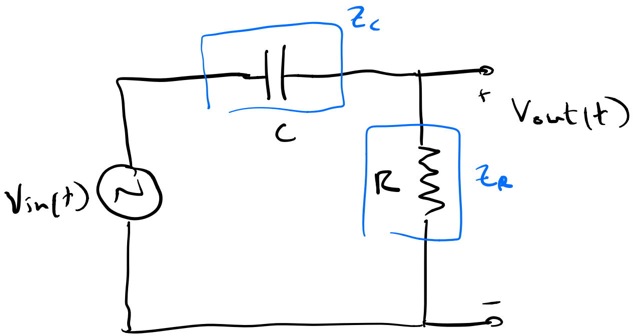




$$\frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = H(\omega)$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \left|\frac{1}{1 + j\omega RC}\right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

What about this one?



$$\frac{V_{out}(\omega)}{V_{in}(\omega)} = H(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

magnitude: $|H(\omega)| = \dfrac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$

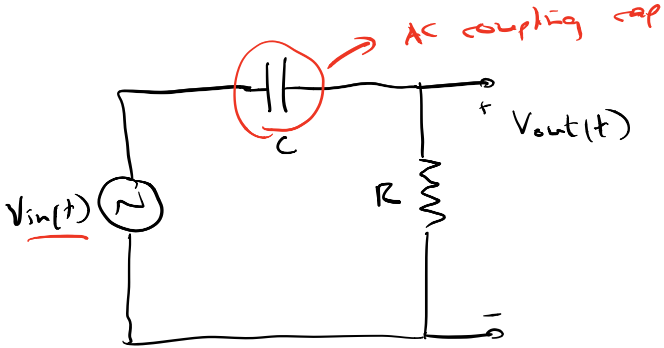
$$\phi(\omega) = \tan^{-1}(\omega RC)$$

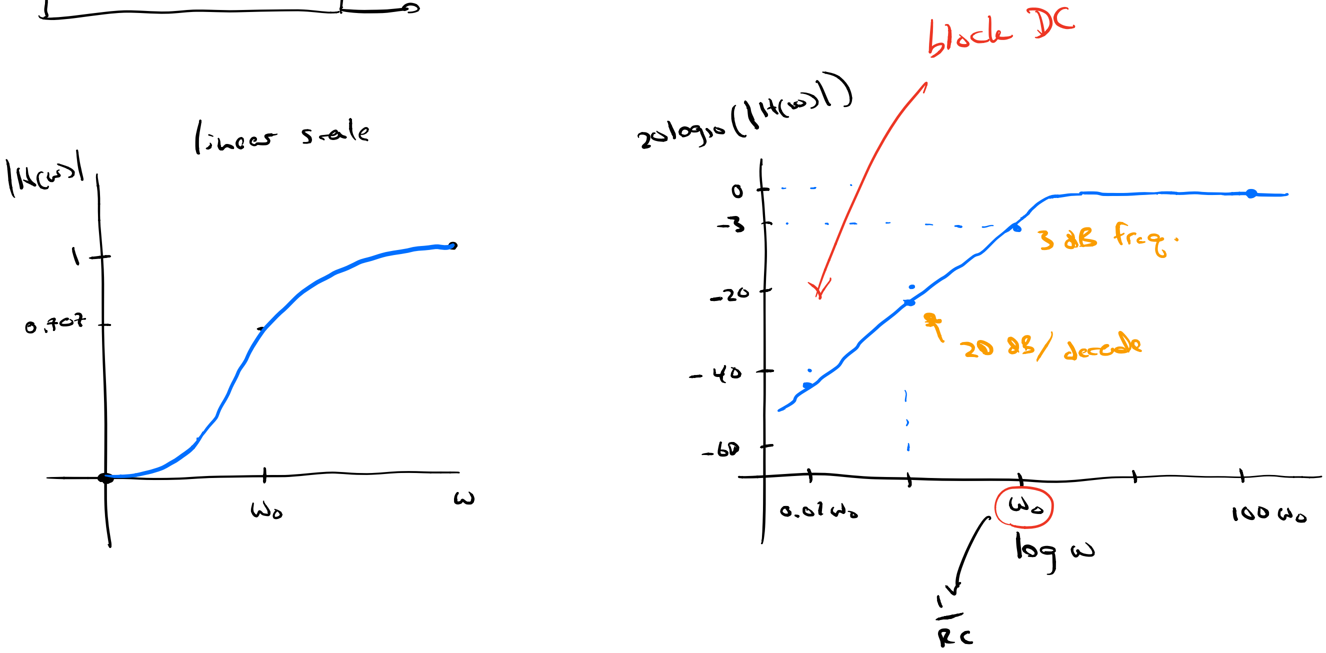
at $\omega = 0$: $|H(\omega)| = 0$

at $\omega \to \infty$: $|H(\omega)| = 1$

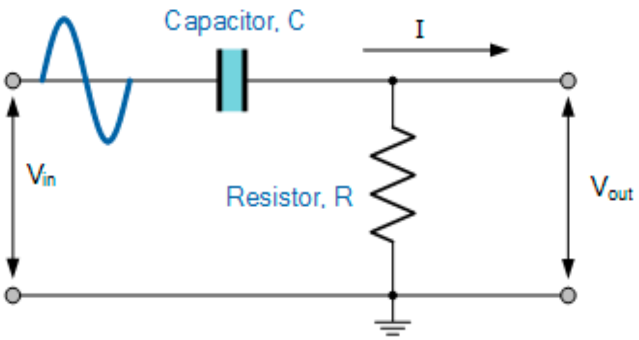
at $\omega = 1/RC$: $|H(\omega)| = 1/\sqrt{2}$

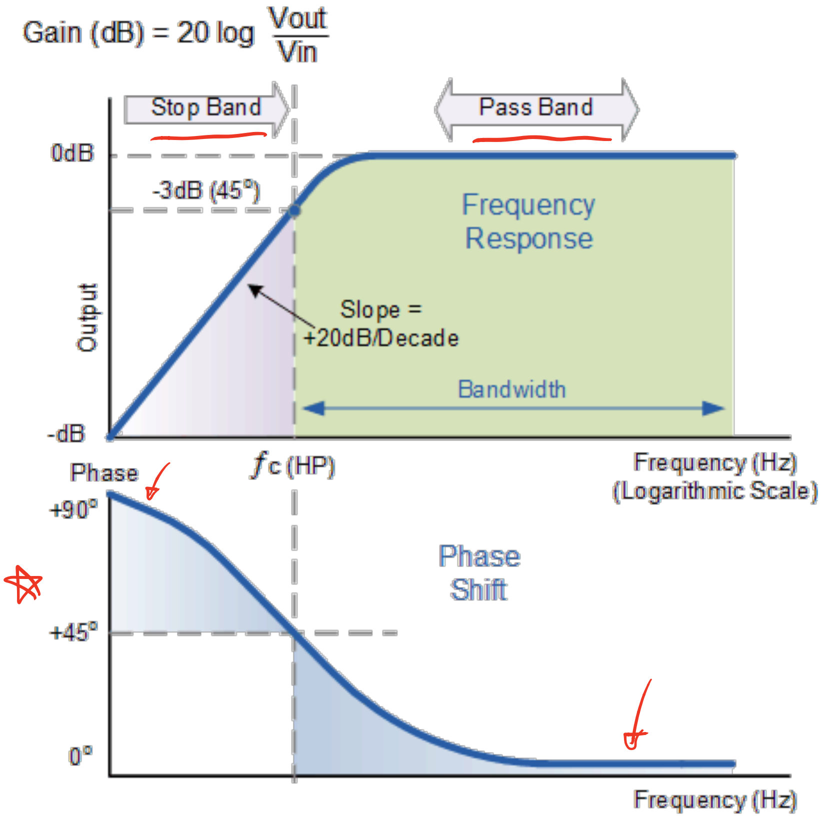
High-pass filter cont.





High-pass filter

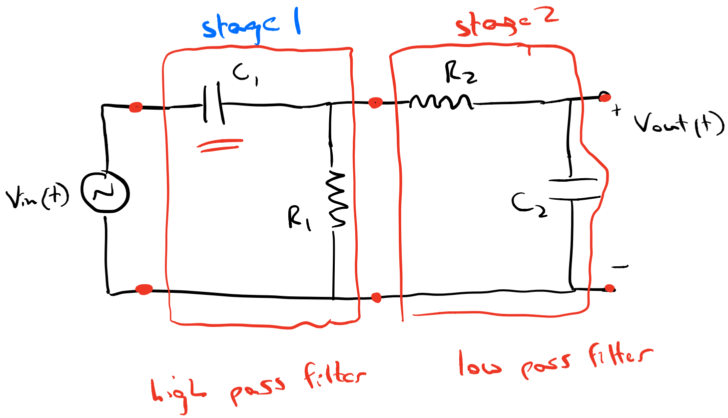




$$\frac{V_{OUT}}{V_{IN}} = H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

$$\left|\frac{V_{OUT}}{V_{IN}}\right| = \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$$

What about this one??



$$|H(\omega)| = |H_1(\omega)| \cdot |H_2(\omega)|$$

$$|H(\omega)| = \frac{\omega R_1 C_1}{\sqrt{1+\omega^2 R_1^2 C_1^2}} \cdot \frac{1}{\sqrt{1+\omega^2 R_2^2 C_2^2}}$$

at $\omega = 0$: $|H(\omega)| = 0$

at $\omega = \infty$: $|H(\omega)| = 0$

at $\omega_{01} = 1/R_1C_1$: $|H(\omega)| = 1/\sqrt{2}$

$\omega_{02} = 1/R_2C_2$: $= 1/\sqrt{2}$

} if R, C chosen carefully